COMPLEMENTS AND CLOSURES ON A SET

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Contents

1. Preliminary results

Definition 1.1. If X is a topological space and $E \subset X$, the *closure* of E in X is the set

$$\overline{E} = \bigcap \{U \subseteq X \mid U \text{ is closed and } E \subseteq U\}.$$

Lemma 1.2. *The operation $A \to \overline{A}$ in a topological space X has the following properties:*

1. $E \subseteq \overline{E}$,
2. $\overline{(\overline{E})} = \overline{E}$,
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$,
4. $\overline{\varnothing} = \varnothing$
5. *E is closed in X if and only if $\overline{E} = E$,*
6. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$,
7. *if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.*

Definition 1.3. If X is a topological space and $E \subseteq X$, the *interior* of E in X is the set

$$\text{Int}\,(E) = \bigcup \{U \subseteq X \mid U \text{ is open and } U \subseteq E\}.$$

Lemma 1.4. *The interior operation $A \to \text{Int}\,(A)$ in a topological space X has the following properties:*

1. $\text{Int}\,(A) \subseteq A$.
2. $\text{Int}\,(\text{Int}\,(A)) = \text{Int}\,(A)$.
3. $\text{Int}\,(A \cap B) = \text{Int}\,(A) \cap \text{Int}\,(B)$.
4. $\text{Int}\,(X) = X$.
5. *E is open if and only if $\text{Int}\,(E) = E$.*
6. $\text{Int}\,(A) \cup \text{Int}\,(B) \subseteq \text{Int}\,(A \cup B)$,
7. *if $A \subseteq B$, then $\text{Int}\,(A) \subseteq \text{Int}\,(B)$.*

Lemma 1.5. *For any topological space X and subset E of X,*

1. $X \setminus \text{Int}\,(E) = \overline{X \setminus E}$,
2. $X \setminus \overline{E} = \text{Int}\,(X \setminus E)$.

2. Unique subsets obtained

We aim to find the maximum number of unique subsets we can derive from a topological space X by applying complements and closures on a given subset E of X. We provide an upper bound in this section, which we show is indeed the maximum by providing examples in subsequent sections.

Corollary 2.1. *Let A be a subset of a topological space X. If $B \subseteq X$ can be represented as alternating interior and closure operations on the set A, then $\overline{X \setminus B}$ and $\operatorname{Int}(X \setminus B)$ can be represented as the complement of alternating interior and closure operations on the the set A.*

Proof. This is an immediate consequence of Lemma 1.5. □

Note. There is great importance of Corollary 2.1. it tells us that we can focus on generating as many unique sets from taking interior and closure operations. Once we have done so, we can take their complements. Corollary 2.1 tells us that taking closure and interior operations on those complements is unnecessary, since they won't give us any new sets.

Proposition 2.2. *Given any subset E of a topological space X,*

(1) $\operatorname{Int}\left(\overline{\operatorname{Int}\left(\overline{E}\right)}\right) = \operatorname{Int}\left(\overline{E}\right)$,

(2) $\overline{\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right)} = \overline{\operatorname{Int}(E)}$.

Proof. Both (1) and (2) have similar proofs. Hence, we only provide the proof of (1).

The result (1) follows from Lemma 1.2 and Lemma 1.4. We get that

$$\begin{aligned} & \operatorname{Int}\left(\overline{E}\right) \subseteq \overline{E} \\ \implies & \overline{\operatorname{Int}\left(\overline{E}\right)} \subseteq \overline{E} \\ \implies & \operatorname{Int}\left(\overline{\operatorname{Int}\left(\overline{E}\right)}\right) \subseteq \operatorname{Int}\left(\overline{E}\right) \\ & \text{and} \\ & \operatorname{Int}\left(\overline{E}\right) \subseteq \overline{\operatorname{Int}\left(\overline{E}\right)} \\ \implies & \operatorname{Int}\left(\overline{E}\right) \subseteq \operatorname{Int}\left(\overline{\operatorname{Int}\left(\overline{E}\right)}\right). \end{aligned}$$

Thus, equality follows (proving (1)). □

Proposition 2.3. *Given any subset E of a topological space X,*

$$\operatorname{Int}(E) \subseteq \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \subseteq \operatorname{Int}\left(\overline{E}\right), \overline{\operatorname{Int}(E)} \subseteq \overline{\operatorname{Int}\left(\overline{E}\right)} \subseteq \overline{E}.$$

Proof. Omitted. Similar proof to Proposition 2.2. □

Corollary 2.4. *Let E be a subset of a topological space X. Then there are at most 7 unique sets, including E, obtained by taking interior and closure operations on E.*

Proof. Our candidates for the 7 sets are E and those found in the subset chain in Proposition 2.3. Let $\mathscr{A}$ be the collection of these sets. Given any $A \in \mathscr{A}$, it is clear that $\operatorname{Int}(A), \overline{A} \in \mathscr{A}$. Thus, there are at most 7 unique sets obtained by taking interior and closure operations on E. □

Note. Using Corollary 2.1 and Corollary 2.4, we get that there is at most 14 unique subsets we can obtain using closures/complements.

3. Minimum number of points

In this section we focus on finite topological spaces. We aim to find the minimum number of points in a topological space required to have an example. Moreover, we wish to find the number of points the subset contains in such an example.

Example 3.1. We provide a finite example with symmetry that demonstrates 14 is the true maximum (not just an upper bound).

Let X be a topological space with 7 distinct points $\{-3, -2, -1, 0, 1, 2, 3\}$. Let

$$\mathscr{B} = \{\{1\}, \{3\}, \{-1, 1\}, \{-2, 2\}, \{-3, 3\}, \{-3, -2, -1, 0, 1, 2, 3\}\}$$

be a base for X. Let $E = \{-3, 1, 2\}$ be a subset of X. Then one can easily obtain that

- $\operatorname{Int}(E) = \{1\}$,
- $\overline{\operatorname{Int}(E)} = \{-1, 0, 1\}$,
- $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) = \{-1, 1\}$,
- $\overline{E} = \{-3, -2, -1, 0, 1, 2\}$,
- $\operatorname{Int}\left(\overline{E}\right) = \{-2, -1, 1, 2\}$,
- $\overline{\operatorname{Int}\left(\overline{E}\right)} = \{-2, -1, 0, 1, 2\}$.

Since each of these are unique sets, this shows that the maximum is 14. Notice the symmetry in our example.

Proposition 3.2. *Let E be a subset of a topological space X. To obtain 14 unique sets from E by applying complements and closure operations on E, X must have at least 7 points.*

Proof. Recall from Proposition 2.3 that

$$\operatorname{Int}(E) \subseteq \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \subseteq \operatorname{Int}\left(\overline{E}\right), \overline{\operatorname{Int}(E)} \subseteq \overline{\operatorname{Int}\left(\overline{E}\right)} \subseteq \overline{E}.$$

For each of these to be unique, it follows that $\operatorname{Int}(E) \neq \varnothing$ and $\overline{E} \neq X$. Moreover, each of these subsets must differ by at least one point. Therefore $\operatorname{Int}(E)$ contains at least one point and following the subset chain we get that $\overline{E}$ has at least five points. This means that X contains at least 6 points. Example 3.1 verifies that using 7 points is possible. Hence, it suffices to show that an example with 6 points does not exist.

For sake to derive a contradiction, suppose X contains precisely 6 points and that E has the desired property. By Proposition 2.3, we know that $\operatorname{Int}(E)$ is a singleton $\{x\}$ for some $x \in X$. We also know that $\overline{E}$ is the complement of a singleton $\{y\}$ for some $y \neq x \in X$. Again using Proposition 2.3, we get that $\operatorname{Int}\left(\overline{E}\right)$ and $\overline{\operatorname{Int}(E)}$ both contain three points. More than this, they are subsets of $\overline{\operatorname{Int}\left(\overline{E}\right)}$ which has four points. Hence, $\operatorname{Int}\left(\overline{E}\right)$ and $\overline{\operatorname{Int}(E)}$ share only two points. Consequently,

$$\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)} = \{z\}$$

for some isolated point $z \in X$. Since $x \in \overline{\operatorname{Int}(E)}$, it is clear $z \neq x$. Moreover, we know that $\operatorname{Int}\left(\overline{E}\right) \subseteq \overline{E} = X \setminus \{y\}$ so that $z \neq y$.

We now have three isolated points x, y and z. We know that $x \in E$ and that $\operatorname{Int}(E) = \{x\}$. So $y, z \notin E$, for otherwise we can obtain a larger open set contained in E. But then $X \setminus \{y, z\}$ is a closed set which contains E and so

$$\overline{E} = X \setminus \{y\} \subseteq X \setminus \{y, z\},$$

a clear contradiction. □

Proposition 3.3. *Let E be a subset of a topological space X, where X has 7 distinct points. If we can obtain 14 unique sets from E by applying complements and closure operations on E, then E either has 3 or 4 points.*

Proof. Clearly E must not have a single point, for otherwise consider $\operatorname{Int}(E)$ (either E is open, or $\operatorname{Int}(E)$ is empty). Hence, E does not have 6 points (consider its complement). Suppose E has 2 points, say x and y. Then w.l.o.g., $\overline{\operatorname{Int}(E)} = \{x\}$. By Proposition 4.1, $y \in \operatorname{Int}\left(\overline{E}\right)$. Hence, $E \subseteq \operatorname{Int}\left(\overline{E}\right)$. That is, $\overline{E} \subseteq \overline{\operatorname{Int}\left(\overline{E}\right)}$, an obvious contradiction (and so E does not have 5 points working with its complement). Result easily follows. □

4. Existence and connectedness

In this section we discuss when an example exists, and some simple consequences of what happens when one does.

In light of the proof given in Proposition 3.2, we provide the following result.

Proposition 4.1. *Let E be a subset of a topological space X. If we can obtain 14 unique sets by applying complements and closures on E, then there exist $x \neq y \in \operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)}$ such that $x \notin E$ and $y \in E$.*

Proof. Suppose that we can obtain 14 unique sets by applying complements and closures on E. By Proposition 2.3 we get

$$\operatorname{Int}(E) \subseteq \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \subseteq \operatorname{Int}\left(\overline{E}\right), \overline{\operatorname{Int}(E)} \subseteq \overline{\operatorname{Int}\left(\overline{E}\right)} \subseteq \overline{E}.$$

We start by showing $\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)}$ must be non-empty. For sake to derive a contradiction, suppose that $\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)} = \varnothing$. That is to say, $\operatorname{Int}\left(\overline{E}\right) \subseteq \overline{\operatorname{Int}(E)}$. But then $\overline{\operatorname{Int}\left(\overline{E}\right)} \subseteq \overline{\operatorname{Int}(E)}$. By the subset chain above (from Proposition 2.3), we know that $\overline{\operatorname{Int}(E)} \subseteq \overline{\operatorname{Int}\left(\overline{E}\right)}$ such that we have equality. This clearly contradicts us being able to obtain 14 unique sets. Hence, $\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)}$ must be non-empty.

Since $\operatorname{Int}(E) \subseteq \overline{\operatorname{Int}(E)}$, it follows that $\operatorname{Int}(E)$ and $\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)}$ are disjoint. Since $\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)}$ is open, there must exist some $x \in \operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)}$ such that $x \notin E$. For otherwise we would obtain the contradiction

$$\operatorname{Int}(E) \neq \operatorname{Int}(E) \cup \left(\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)}\right) \subseteq E.$$

For sake to derive a contradiction, suppose that $\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)}$ and E are disjoint. We know that

$$\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)} \subseteq \overline{E}$$

by Proposition 2.3. That is to say, $\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)}$ and $X \setminus \overline{E}$ are disjoint open sets. But then their union is an open set larger than $X \setminus \overline{E}$ which contains no points of E. This is a clear contradiction. Thus, there exists $y \neq x \in \operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)}$ such that $y \in E$. □

Note. We might also wish to show similar result (or that it does not hold) for $X \setminus \operatorname{Int}\left(\overline{E}\right) \cap \overline{\operatorname{Int}(E)}$.

Corollary 4.2. *Let E be a subset of a topological space X. If each set in the chain*

$$\operatorname{Int}(E) \subseteq \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \subseteq \operatorname{Int}\left(\overline{E}\right), \overline{\operatorname{Int}(E)} \subseteq \overline{\operatorname{Int}\left(\overline{E}\right)} \subseteq \overline{E}$$

are unique, then we can obtain 14 unique subsets by applying complements and closures on E.

Proof. We need only prove that E is different from each set in the subset chain. We will provide a proof by contradiction. To this end, suppose that E is equal to one of the sets in the subset chain. Then E is either open or closed. Then $\operatorname{Int}(E) = E$ or $\overline{E} = E$ for each of the respective cases. But this clearly contradicts each of the sets in the subset chain being unique. For we would otherwise have $\overline{\operatorname{Int}(E)} = \overline{E}$ or $\operatorname{Int}\left(\overline{E}\right) = \operatorname{Int}(E)$. $\square$

Proposition 4.3. *Let E be a subset of a connected topological space X with non-empty interior. Then we can obtain 14 unique sets by applying complements and closures on E if and only if $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \notin \left\{\operatorname{Int}(E), \operatorname{Int}\left(\overline{E}\right)\right\}$ and $\overline{\operatorname{Int}\left(\overline{E}\right)} \neq \overline{E}$.*

Proof. ($\Longrightarrow$) It is clear that obtaining 14 unique sets by applying complements and closures on E is gives us $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \notin \left\{\operatorname{Int}(E), \operatorname{Int}\left(\overline{E}\right)\right\}$ and $\overline{\operatorname{Int}\left(\overline{E}\right)} \neq \overline{E}$.

($\Longleftarrow$) Suppose $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \notin \left\{\operatorname{Int}(E), \operatorname{Int}\left(\overline{E}\right)\right\}$ and $\overline{\operatorname{Int}\left(\overline{E}\right)} \neq \overline{E}$. By Proposition 2.3,

$$\operatorname{Int}(E) \subseteq \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \subseteq \operatorname{Int}\left(\overline{E}\right), \overline{\operatorname{Int}(E)} \subseteq \overline{\operatorname{Int}\left(\overline{E}\right)} \subseteq \overline{E}.$$

By the subset chain and that $\operatorname{Int}(E)$ is non-empty, it follows that every set in the subset chain is non-empty. We aim to show that each of these are unique. By Corollary 4.2, this is sufficient.

We start by showing $\overline{E}, \overline{\operatorname{Int}(E)}$ and $\overline{\operatorname{Int}\left(\overline{E}\right)}$ are different from one another. If $\overline{E} = \overline{\operatorname{Int}(E)}$, then taking the interior operator on both sides gives us that $\operatorname{Int}\left(\overline{E}\right) = \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right)$ (a contradiction). If $\overline{\operatorname{Int}(E)} = \overline{\operatorname{Int}\left(\overline{E}\right)}$, then $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) = \operatorname{Int}\left(\overline{E}\right)$ (a contradiction). The rest follows from our assumptions.

Now, we show $\operatorname{Int}(E), \operatorname{Int}\left(\overline{E}\right)$ and $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right)$ are different from one another. If $\operatorname{Int}(E) = \operatorname{Int}\left(\overline{E}\right)$, then $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) = \operatorname{Int}\left(\overline{E}\right)$ (a contradiction). The rest follows from our assumptions.

By connectedness of X, the rest easily follows. $\square$

Note. While it seems, with what we have so far, we must have that $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \neq \operatorname{Int}(E)$ and $\overline{\operatorname{Int}\left(\overline{E}\right)} \neq \overline{E}$, we did not have to choose $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \neq \operatorname{Int}\left(\overline{E}\right)$. This is because $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \neq \operatorname{Int}\left(\overline{E}\right)$ if and only if $\overline{\operatorname{Int}(E)} \neq \overline{\operatorname{Int}\left(\overline{E}\right)}$ (and so we could have chosen this instead). We remove both of these conditions in the following result.

Corollary 4.4. *Let E be a subset of a connected topological space X with non-empty interior. Then we can obtain 14 unique sets by applying complements and closures on E if and only if $\operatorname{Int}(E) \neq \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right)$, $\overline{E} \neq \overline{\operatorname{Int}\left(\overline{E}\right)}$ and $\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)} \neq \varnothing$.*

Proof. ($\Longrightarrow$) Follows from Proposition 4.1 and Proposition 4.3.

($\Longleftarrow$) By Proposition 4.3, it suffices to show that $\operatorname{Int}\left(\overline{E}\right) \neq \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right)$. Suppose that they were equal. Then

$$\varnothing \neq \operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)} = \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \cap X \setminus \overline{\operatorname{Int}(E)} = \varnothing,$$

a contradiction. $\square$

Note. Perhaps by removing connectedness, we can change $\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)} \neq \varnothing$ to that seen in Proposition 4.1.

5. Open and closed sets

In this section we will answer the same question for when E is either open or closed (or both), and provide some other basic results for the interior/closure operators.

Proposition 5.1. *E is an open set if and only if*

$$E = \operatorname{Int}(E) \subseteq \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) = \operatorname{Int}\left(\overline{E}\right) \subseteq \overline{\operatorname{Int}(E)} = \overline{\operatorname{Int}\left(\overline{E}\right)} = \overline{E}.$$

Proof. Obviously if $\operatorname{Int}(E) = E$ then E is open. Now, suppose E is open. Then

$$E \subseteq \operatorname{Int}\left(\overline{E}\right)$$

which implies

$$\overline{E} \subseteq \overline{\operatorname{Int}\left(\overline{E}\right)}$$

such that they are equal (refer to Proposition 2.3 in this proof). Moreover $\overline{E} = \overline{\operatorname{Int}(E)}$ rather trivially. Lastly observe

$$\operatorname{Int}\left(\overline{E}\right) = \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right)$$

so that the result follows. □

Note. We can obtain an upper bound of 6 from this, which we show is the maximum in the following example.

Example 5.2. Merely take the same topology from Example 3.1 and instead start with $E = \{1\}$.

Proposition 5.3. *E is a closed set if and only if*

$$\operatorname{Int}(E) = \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) = \operatorname{Int}\left(\overline{E}\right) \subseteq \overline{\operatorname{Int}(E)} = \overline{\operatorname{Int}\left(\overline{E}\right)} \subseteq \overline{E} = E.$$

Proof. Obviously if $\overline{E} = E$ then E is closed. Now, suppose E is closed. Then

$$\overline{\operatorname{Int}(E)} \subseteq E$$

which implies

$$\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \subseteq \operatorname{Int}(E)$$

so that equality holds (refer to Proposition 4.2 in this proof). Moreover $\operatorname{Int}(E) = \operatorname{Int}\left(\overline{E}\right)$ rather trivially. Lastly, observe that

$$\overline{\operatorname{Int}(E)} = \overline{\operatorname{Int}\left(\overline{E}\right)}$$

so that the result follows. □

Note. Again we obtain a maximum of 6, but this should be no surprise. Rather trivially, if E is clopen then the maximum is 2 (itself and its complement).

A more interesting question is now to ask the following: suppose we have some open set (closed set) E of a topological space X which is not closed (open). Then what is the minimum number of unique subsets we can obtain? We know the maximum is 6 and that it must be even (so candidates are $2, 4, 6$). But we can clearly rule out 2. Let $X = \mathbb{R}$ and $E = (0,1)$ (or $[0,1]$). This is an example of obtaining 4, so we have shown that the minimum is 4 and the maximum 6.

Corollary 5.4. *Let E be a subset of a topological space X. Then* $\operatorname{Int}(E)$ *attains* 6 *unique sets and* $\overline{E}$ *attains* 6 *unique sets if and only if E attains* 14 *unique sets.*

Proof. Suppose that $\operatorname{Int}(E)$ attains 6 unique sets and $\overline{E}$ attains 6 unique sets. Then

$$\operatorname{Int}(E) \subsetneq \operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \subsetneq \overline{\operatorname{Int}(E)}$$

and

$$\operatorname{Int}\left(\overline{E}\right) \subsetneq \overline{\operatorname{Int}\left(\overline{E}\right)} \subsetneq \overline{E}.$$

By Proposition 2.3, it suffices to show that $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \neq \operatorname{Int}\left(\overline{E}\right)$ and that $\overline{\operatorname{Int}(E)} \neq \operatorname{Int}\left(\overline{E}\right), \overline{\operatorname{Int}\left(\overline{E}\right)}$. Since $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) \neq \operatorname{Int}\left(\overline{E}\right)$ if and only if $\overline{\operatorname{Int}(E)} \neq \overline{\operatorname{Int}\left(\overline{E}\right)}$, we need only show that $\overline{\operatorname{Int}(E)} \neq \operatorname{Int}\left(\overline{E}\right), \overline{\operatorname{Int}\left(\overline{E}\right)}$. If $\overline{\operatorname{Int}(E)} = \operatorname{Int}\left(\overline{E}\right)$, then $\overline{\operatorname{Int}(E)} = \overline{\operatorname{Int}\left(\overline{E}\right)}$. So we need only show that $\overline{\operatorname{Int}(E)} \neq \overline{\operatorname{Int}\left(\overline{E}\right)}$.

For sake to derive a contradiction, suppose that $\overline{\operatorname{Int}(E)} = \overline{\operatorname{Int}\left(\overline{E}\right)}$. Then

$$\operatorname{Int}\left(\overline{E}\right) \subsetneq \overline{\operatorname{Int}(E)}.$$

That is to say, $\operatorname{Int}\left(\overline{E}\right) \cap X \setminus \overline{\operatorname{Int}(E)} = \varnothing$. ☐

Proposition 5.5. *Given any subset E of a topological space $\langle X, \mathscr{T} \rangle$ with $E, X \setminus E \notin \mathscr{T}$, $\operatorname{Int}\left(\overline{E}\right) \neq \varnothing$ and $\operatorname{Int}\left(\overline{X \setminus E}\right) \neq \varnothing$, we have*

$$\operatorname{Int}\left(\overline{E}\right) \cap \operatorname{Int}\left(\overline{X \setminus E}\right) \neq \varnothing.$$

Proof. Let $x \in \operatorname{Int}\left(\overline{E}\right)$. Then there exists some open nhood U of x contained in $\overline{E}$. Since E is neither open or closed, we have that

$$\overline{E} \cap \overline{X \setminus E} \neq \varnothing.$$

Now, since

$$\operatorname{Int}\left(\overline{E} \cap \overline{X \setminus E}\right) = \operatorname{Int}\left(\overline{E}\right) \cap \operatorname{Int}\left(\overline{X \setminus E}\right),$$

we must prove that $C = \overline{E} \cap \overline{X \setminus E}$ has non-empty interior. Let $x \in C$. If every nhood of x meets $X \setminus C$, then ☐

6. Metric spaces

Example 6.1. Consider the metric space $(\mathbb{R}, d)$, where d is the usual metric on $\mathbb{R}$. Let $F = \{x \in \mathbb{Q} \mid 2 \leq x \leq 3\}$ and $U = \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n}\right)$. Then $E = F \cup U \cup \mathbb{N}$ is an example:

1. $\operatorname{Int}(E) = U$,
2. $\overline{\operatorname{Int}(E)} = [0, 1]$,
3. $\operatorname{Int}\left(\overline{\operatorname{Int}(E)}\right) = (0, 1)$,
4. $\overline{E} = [2, 3] \cup [0, 1] \cup \mathbb{N}$,
5. $\operatorname{Int}\left(\overline{E}\right) = (2, 3) \cup (0, 1)$,
6. $\overline{\operatorname{Int}\left(\overline{E}\right)} = [2, 3] \cup [0, 1]$.